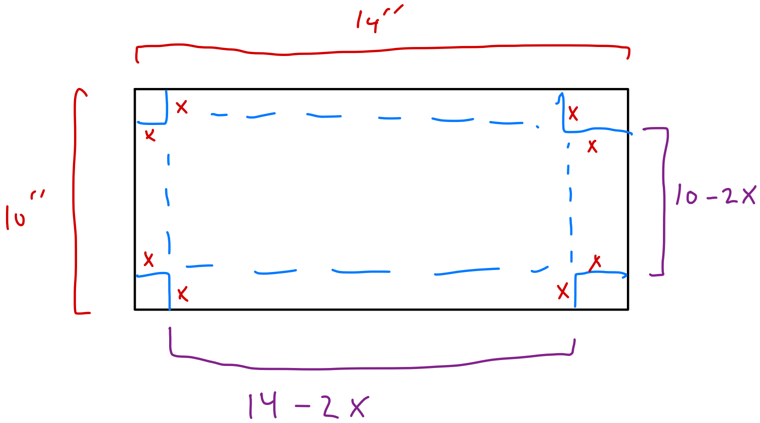
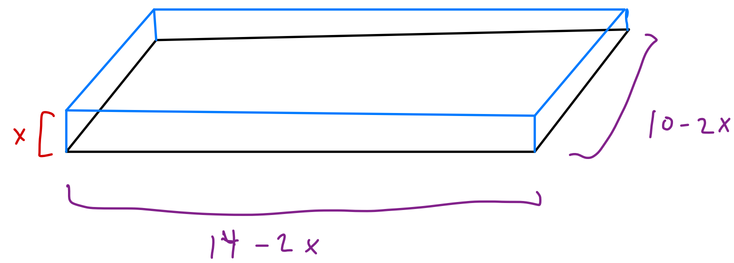
Example. We have a piece of cardboard that is 14 inches by 10 inches, and we are going to cut out the corners and fold up the sides to form a lidless box. Determine dimensions of the box that maximizes volume.





$$V = (\text{length})(\text{width})(\text{height})$$

$$\Rightarrow \quad V = (14-2x)(10-2x)(x)$$

$$\Rightarrow \quad V = [14\cdot 10 - 2x\cdot 10 + 14(-2x) + (-2x)(-2x)]\, x$$

$$\Rightarrow \quad V = [140 - 48x + 4x^2]\, x$$

$$\Rightarrow \quad V = 140x - 48x^2 + 4x^3$$

$$V' = 140 - 48\cdot 2\cdot x + 4\cdot 3\cdot x^2$$

$$= 140 - 96x + 12x^2$$

quadratic formula to determine solutions to $140 - 96x + 12x^2 = 0$

$$\Rightarrow \quad x = \underbrace{4 - \sqrt{\tfrac{13}{3}}}_{\approx 1.9183},\ \underbrace{4 + \sqrt{\tfrac{13}{3}}}_{6.0817}$$

$\Rightarrow$ doesn't make sense as width

$\approx 10 - 2\cdot 6.0817 < 0$

Verify max at $x = 4 - \sqrt{\frac{13}{3}}$

Recall $V' = 140 - 96x + 12x^2$

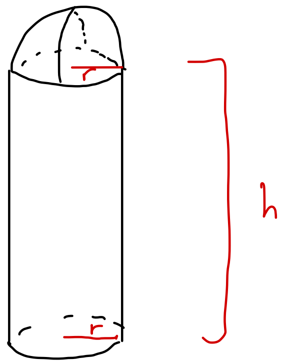
$V'' = -96 + 24x \Rightarrow V''\left(4 - \sqrt{\frac{13}{3}}\right) = -96 + 24\left(4 - \sqrt{\frac{13}{3}}\right) < 0$

By 2nd derivative test, $x = 4 - \sqrt{\frac{13}{3}}$ is the location of a max.

$\Rightarrow$ optimal dimensions are $14 - 2\left(4 - \sqrt{\frac{13}{3}}\right)''$ by $10 - 2\left(4 - \sqrt{\frac{13}{3}}\right)''$ by $\left(4 - \sqrt{\frac{13}{3}}\right)''$

(approx) 10.16'' by 6.16'' by 1.983''

Example. A silo with a cylindrical base is being mounted with a hemisphere. The material to make the hemisphere costs \$30 per square meter. The material to make the cylindrical sidewall is \$10 per square meter. The silo should have a fixed volume of 5000 cubic meters. Find the dimensions of the silo that minimizes the cost to build it.



Volume of cylinder $= \pi r^2 h$

Volume of hemisphere $= \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$
$= \frac{2}{3}\pi r^3$

$V_{silo} = \pi r^2 h + \frac{2}{3}\pi r^3 = 5000$

solve for h $\Rightarrow$ $\pi r^2 h = 5000 - \frac{2}{3}\pi r^3$

$h = \dfrac{5000 - \frac{2}{3}\pi r^3}{\pi r^2}$

$\Rightarrow h = \frac{5000}{\pi}\cdot r^{-2} - \frac{2}{3} r$

$SA_{cylinder} = (2\pi r)(h)$ m^2

$SA_{hemisphere} = \frac{1}{2}(4\pi r^2)$ m^2

$\Rightarrow cost_{cylinder} = \left(10 \frac{\text{dollars}}{m^2}\right)\left(2\pi r h \ m^2\right)$

$= 20\pi r h$ dollars

$Cost_{hemisphere} = \left(30 \frac{\text{dollars}}{m^2}\right)\left(2\pi r^2 \ m^2\right)$

$= 60\pi r^2$ dollars

total cost $= 20\pi r h + 60\pi r^2$ dollars

$\Rightarrow C = 20\pi r\left(\frac{5000}{\pi}\cdot r^{-2} - \frac{2}{3} r\right) + 60\pi r^2$

$\Rightarrow C = 100{,}000\, r^{-1} - \frac{40\pi}{3} r^2 + 60\pi r^2$

$C' = -100{,}000 r^{-2} - \frac{40\pi}{3}\cdot 2\cdot r + 120\pi r$

$= -100{,}000 r^{-2} - \frac{80\pi}{3}\cdot r + \frac{360\pi}{3} r = -100{,}000 r^{-2} + \frac{280\pi}{3}\cdot r$

$\Rightarrow C' = -\dfrac{100{,}000}{r^2} + \dfrac{280\pi}{3}\cdot r$

Note, C' DNE if $r = 0$, but this is nonsensical for our problem

$$C' = \frac{-100{,}000}{r^2} + \frac{280\pi}{3}\cdot r = 0$$

$$\Rightarrow \frac{280\pi}{3}\cdot r = \frac{100{,}000}{r^2}$$

$$\Rightarrow \frac{280\pi}{3} r^3 = 100{,}000$$

$$\Rightarrow r^3 = \frac{100{,}000}{\left(\frac{280\pi}{3}\right)}$$

$$r = \left(\frac{100{,}000}{\left(\frac{280\pi}{3}\right)}\right)^{\frac{1}{3}} \approx 6{,}986$$

sign chart for $C' = -\frac{100000}{r^2} + \frac{280\pi}{3}\cdot r$

$$\begin{array}{ccc} 0 & \left(\frac{100000}{\frac{280\pi}{3}}\right)^{\frac{1}{3}} & \\ - & & + \end{array}$$

try $r=1$

$C' = -\frac{100000}{1} + \frac{280\pi}{3}\cdot 1$

< 0

try $r = 100000$

$C' = -\frac{100000}{100000} + \frac{280\pi}{3}\cdot 100000$

> 0

By 1st derivative test, $r = \left(\frac{100{,}000}{\frac{280\pi}{3}}\right)^{\frac{1}{3}}$ is location of min

So the optimal radius is $r = \left(\frac{100000}{\frac{280\pi}{3}}\right)^{\frac{1}{3}} \approx 6{,}986$ m

and height $h = \frac{5000}{\pi} r^{-2} - \frac{2}{3} r = \frac{5000}{\pi}\left(\left(\frac{100000}{\frac{280\pi}{3}}\right)^{\frac{1}{3}}\right)^{-2} - \frac{2}{3}\left(\frac{100000}{\frac{280\pi}{3}}\right)^{\frac{1}{3}}$

≈ 27.91 m